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Best Mean Approximation by Splines Satisfying Generalized Convexity Constraints

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A characterization of the best L_1-approximation to a continuous function by classes of fixed-knot polynomial splines which satisfy generalized convexity constraints is presented and uniqueness is shown. Included is the possibility of specifying the positivity, monotonicity, or convexity of the class. The proof of uniqueness uses recently developed results for Hermite–Birkhoff interpolation by splines.

INTRODUCTION

The concept of monotone approximation by polynomials was introduced by Shisha [13], and has been further studied by many authors. Lorentz [5] demonstrated uniqueness, in general, for best approximation by monotone polynomials in the uniform and L_1-norm. Roulier and Taylor [10] generalized this monotonicity constraint to include more general restrictions on the range of derivatives. An excellent survey to this and subsequent work concerning uniform approximation with constraints can be found in Chalmers and Taylor [3]. If all the restrictions on derivatives are nonnegativity or nonpositivity, we will call them generalized convexity constraints.

Classes of polynomial splines with fixed knots satisfying generalized convexity constraints and other inequality-type constraints were introduced and studied in the author's thesis [7], with some of the results appearing in [8]. Best uniform approximations were characterized and partial uniqueness was established. This paper continues the study of such constrained splines by considering the L_1-norm. A characterization is given which is a special case of a more general result found in Rozema and Smith [11]. Further uniqueness is established, too.

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Uniqueness for best L_1-approximation by certain unconstrained splines with fixed knots has previously been demonstrated by Carroll and Braess [2] and Galkin [4]. Thus we generalize their results by allowing constraints. Pinkus [9] has considered best one-sided L^1-approximation by splines to a differentiable function. We indicate in the last section how the methods of this paper can be used to handle one-sided approximation as well as more general restrictions on the range of derivatives. Only the somewhat simpler case of generalized convexity constraints is presented in detail.

We will use results for Hermite–Birkhoff interpolation by splines which were developed by the author [7, 8]. For completeness the required theory is briefly reviewed in the following section together with certain rather technical interpolation lemmas which are needed.

1. Hermite–Birkhoff Interpolation by Splines

Suppose $-\infty < a = \xi_0 < \xi_1 < \cdots < \xi_q < \xi_{q+1} = b < \infty$ and integers R_ν with $0 < R_\nu \leqslant m$, $\nu = 1,..., q$, are given. Let $\mathscr{S}_p^m = \mathscr{S}_p^m(\{\xi_\nu\}_1^q ; \{R_\nu\}_1^q)$ denote the space of polynomial spline functions of order m with fixed knots $\{\xi_\nu\}_1^q$, each with multiplicity R_ν, respectively, where $p = \sum_{\nu=1}^q R_\nu$. Thus $g \in \mathscr{S}_p^m$ is piecewise a polynomial of degree at most $m - 1$ with $g^{(j)}$ discontinuous only at a knot ξ_ν where $j \geqslant m - R_\nu$. We adopt the convention that all elements of $\mathscr{S}_p^m$ and all derivatives of elements of $\mathscr{S}_p^m$ are defined everywhere by assuming continuity from the right. Notice that dim $\mathscr{S}_p^m = m + p$.

We review needed facts about Hermite–Birkhoff interpolation (HBI) by polynomial splines.

Let interpolation points

$$X = \{a \leqslant x_1 < x_2 < \cdots < x_k \leqslant b\} \tag{1.1}$$

be given. A matrix

$$E = \{e_{ij}\}, \qquad i = 1,..., k; \quad j = 0, 1,..., m - 1 \tag{1.2}$$

is called a *spline incidence matrix* for X and $\mathscr{S}_p^m$ provided $e_{ij} = 0, \pm 1$, or 2 and $e_{ij} = -1$ or 2 only if $x_i = \xi_\nu$ for some ν and $j \geqslant m - R_\nu$. The HBI problem defined by $(E, X, \mathscr{S}_p^m)$ is:

Given any values $\{\gamma_{ij}: e_{ij} = 1 \text{ or } 2\}$ and $\{\gamma_{ij}^-: e_{ij} = -1 \text{ or } 2\}$, find $g \in \mathscr{S}_p^m$ with

$$g^{(j)}(x_i) = \gamma_{ij} \qquad \text{whenever} \qquad e_{ij} = 1 \text{ or } 2, \tag{1.3}$$

$$g^{(j)}(x_i-) = \gamma_{ij}^- \qquad \text{whenever} \qquad e_{ij} = -1 \text{ or } 2. \tag{1.4}$$

As in [7, 8], when we display such a matrix E, we indicate the relationship between the interpolation points X and the knots of the spline space $\mathscr{S}_p^m$ by drawing the following lines:

(i) If $x_i < \xi_\nu < x_{i+1}$, we draw a solid line between the ith and $(i+1)$th rows extending from the $(m - R_\nu)$th column to the $(m-1)$th column. If more than one knot lies between x_i and x_{i+1}, then draw several lines.

(ii) If $x_i = \xi_\nu$, we enclose in a box the entries in the ith row from the $(m - R_\nu)$th column to the $(m-1)$th column.

Thus an entry of E may be -1 or 2 only if it is boxed.

Define

$$|E| = \sum_{i,j} |e_{ij}|. \tag{1.5}$$

We say $(E, X, \mathscr{S}_p^m)$ is *full* when $\|E\| = \dim \mathscr{S}_p^m = m + p$. If $(E, X, \mathscr{S}_p^m)$ has a unique solution for any given data values or, equivalently, if the only solution to the homogeneous problem is the zero spline, the problem is called *poised*. Obviously $(E, X, \mathscr{S}_p^m)$ must be full for this to happen. When $\|E\| \leqslant m + p$, we say $(E, X, \mathscr{S}_p^m)$ is *quasi-poised* if the dimension of the solution space for the homogeneous problem is exactly $m + p - |E|$.

We now define what are essentially submatrices of E. For $n = 0, 1, \ldots, m-1$ and $0 \leqslant l < s \leqslant q + 1$, let $k_1 = \min\{i: \xi_l \leqslant x_i\}$, $k_2 = \max\{i: x_i \leqslant \xi_s\}$, and

$$E(n: l, s) = \{e_{ij}^*\}, \qquad i = k_1, \ldots, k_2;\ j = n, \ldots, m-1, \tag{1.6}$$

where

$$\begin{aligned} e_{ij}^* &= 1, && \text{if } i = k_1,\ x_i = \xi_l, \text{ and } e_{ij} = 1 \text{ or } 2, \\ &= e_{ij}, && \text{if } x_i \in (\xi_l, \xi_s) \text{ or if } i = k_2,\ x_i = \xi_s, \text{ and } j < m - R_s, \\ &= 1, && \text{if } i = k_2,\ x_i = \xi_s, \text{ and } e_{ij} = -1 \text{ or } 2, \\ &= 0, && \text{otherwise.} \end{aligned} \tag{1.7}$$

By a simple dimension argument, it is easy to see that the following, called the *local Polya conditions* (LPC) for $(E, X, \mathscr{S}_p^m)$, are necessary for quasi-poisedness:

$$\|E(n: l, s)\| \leqslant m - n + p(n: l, s), \qquad \text{for all } n = 0, 1, \ldots, m-1;\ 0 \leqslant l < s \leqslant q + 1, \tag{1.8}$$

where

$$\begin{aligned} p(n: l, s) &= \sum_{\nu=l+1}^{s-1} \min[R_\nu, m - n], && \text{if } l + 1 < s, \\ &= 0, && \text{if } l + 1 = s. \end{aligned} \tag{1.9}$$

It is also easily verified that all of the LPC are satisfied if we have that

$$\|E(n: l, s)\| \leqslant m - n + p(n: l, s), \qquad \text{for all } \{(n, l, s): R_\nu < m - n, \text{ when } l < \nu < s\}. \tag{1.10}$$

In particular when $(E, X, \mathscr{S}_p^m)$ is full, the LPC imply that

$$|E| - |E(0: l, q-1)| \geqslant \sum_{\nu=1}^{l} R_\nu, \qquad l = 1,\dots, q. \tag{1.11}$$

LEMMA 1.1. *If* $|E(0: l, s)| = m - p(0: l, s)$ *for some* $0 < l \leqslant s \leqslant q-1$ *or* $0 \leqslant l < s < q-1$, *then* $(E, X, \mathscr{S}_p^m)$ *can be split vertically into two or three HBI problems, each defined on a spline space of order still* m *but with fewer knots than* $\mathscr{S}_p^m$. *The "central" one of the decomposed problems has incidence matrix* $E(0: l, s)$.

LEMMA 1.2. *If* $x_i = \xi_\nu$ *for some* i *and* ν, *and* $e_{ij} = 1$ *for all* $j = 0, 1,\dots, m - R_\nu - 1$, *or if* $R_\nu = m$ *for some* ν, *then* $(E, X, \mathscr{S}_p^m)$ *can also be split vertically into two HBI problems considering fewer knots.* $E(0: 0, \nu)$ *and* $E(0: \nu, q+1)$ *will be the incidence matrices for these two smaller problems.*

We further note that the above decompositions preserve the LPC and that if the original problem has a full matrix, then so do all of the smaller problems. Quasi-poisedness of $(E, X, \mathscr{S}_p^m)$ is equivalent to quasi-poisedness of all of the split problems. Similar decompositions have been noted by several authors, see [6], for example. The complete details are tedious but not hard and can be found in [7].

Let $(E, X, \mathscr{S}_p^m)$ indicate a given HBI problem. If $x_i \notin \{\xi_\nu\}_1^q$, then we say that we have a *regular sequence* beginning with e_{ij} of order μ when $e_{ij} = e_{i,j-1} = \cdots = e_{i,j-\mu-1} = 1$ with $e_{i,j-1} = 0$ and $e_{i,j-\mu} = 0$ if either is defined. Also if $x_i = \xi_\nu$, then we say that we have a *regular sequence* beginning with e_{ij} of order μ when $e_{ij} = e_{i,j-1} = \cdots = e_{i,j-\mu-1} = 1$ with $j - \mu \leqslant m - R_\nu$, $e_{i,j-1} = 0$ and $e_{i,j+\mu} = 0$ if either is defined. Further a regular sequence $e_{ij},\dots, e_{i,j-\mu-1}$ is called *strongly regular* if $e_{i,j+\mu}$ is defined, zero, and, in the case where $x_i = \xi_\nu$, $j + \mu < m - R_\nu$. A sequence is *even* if it has even order and *odd* otherwise.

We say that a regular sequence $e_{ij},\dots, e_{i,j-\mu-1}$ is *supported* provided there exist integers i_1, j_1, i_2, j_2 with $i_1 < i < i_2$, $e_{i_1 j_1} = 1$ or 2,

$$j_1 \leqslant \min[j, \{m - R_\nu: x_{i_1} < \xi_\nu < x_i\}], \tag{1.12}$$

$$j_2 < \min[j, \{m - R_\nu: x_i < \xi_\nu < x_{i_2}\}], \tag{1.13}$$

and

$$\begin{aligned} e_{i_2 j_2} &= 1, && \text{if } x_{i_2} \notin \{\xi_\nu\}_1^q, \\ &= 1, && \text{if } x_{i_2} = \xi_\nu \text{ and } j_2 < m - R_\nu, \\ &= 1 \text{ or } 2, && \text{if } x_{i_2} = \xi_\nu \text{ and } j_2 \geqslant m - R_\nu. \end{aligned} \tag{1.14}$$

The problem $(E, X, \mathscr{S}_p^m)$ is called *weakly conservative* (C) if every supported strongly regular sequence is even.

THEOREM 1.1. *Suppose $(E, X, \mathscr{S}_p^m)$ satisfies the LPC and C. Then it is quasi-poised.*

This theorem generalizes the sufficiency theorem of Atkinson and Sharma for HBI by polynomials [1]. The proof can be found in [7, 8].

We shall need the following technical lemmas. All three lemmas concern attempts to add conditions of some sort to a given HBI problem.

LEMMA 1.3. *Suppose $(E, X, \mathscr{S}_p^m)$ satisfies the LPC but when some strongly regular sequence in E is extended to have an additional one to the right giving the matrix $\tilde{E}$, then the LPC are violated. There exists $\xi \notin \{\xi_\nu\}_1^q \cup X$ so that when ξ is added as a simple knot to the spline space, then $(\tilde{E}, X, \mathscr{S}_{p+1}^m(\{\xi_\nu\}_1^q, \xi; \{R_\nu\}_1^q, 1))$ satisfies the LPC.*

Proof. Suppose $e_{i,j-\mu} = \cdots = e_{i,j-1} = 1$ is the strongly regular sequence of E and that e_{ij} is changed from a zero to a one to obtain $\tilde{E}$. Then there exists $\eta \leqslant j$ and $0 \leqslant l < s \leqslant q + 1$ with $x_i \in [\xi_l, \xi_s]$ so that for $\mathscr{S}_p^m$,

$$\begin{aligned} |\tilde{E}(\eta: l, s)| &= |E(\eta: l, s)| + 1 \\ &= m - \eta - p(\eta: l, s) + 1. \end{aligned} \tag{1.15}$$

Without loss of generality assume that (1.15) cannot happen first for any $\tilde{\eta} > \eta$ and secondly with η for any $\tilde{l}$ and $\tilde{s}$ with $[\xi_{\tilde{l}}, \xi_{\tilde{s}}] \subsetneqq [\xi_l, \xi_s]$.

Let

$$\epsilon = \min[(x_i - x_{i-1}), (x_{i+1} - x_i), \{|x_i - \xi_\nu| : \xi_\nu \neq x_i\}]. \tag{1.16}$$

We choose $\xi \in (x_i - \epsilon, x_i + \epsilon)\setminus\{x_i\}$ in such a way that when ξ is added to the knot set, i.e., $\{\xi'_\nu\}_1^{q+1} = \{\xi_\nu\}_1^q \cup \{\xi\}$ properly ordered, $\xi = \xi'_{\tilde{l}}$, $R'_{\tilde{l}} = 1$, and $l < \tilde{l} < s + 1$, we have

$$|\tilde{E}(\eta: l, \tilde{l})| < |\tilde{E}(\eta: l, s - 1)|, \tag{1.17}$$

and

$$|\tilde{E}(\eta: \tilde{l}, s + 1)| < |\tilde{E}(\eta: l, s - 1)|, \tag{1.18}$$

with respect to $\mathscr{S}_{p+1}^m(\{\xi'_\nu\}_1^{q+1}; \{R'_\nu\}_1^{q+1})$.

It is easily seen that this can be done. The proof is completed by checking the various ways $\tilde{E}$ might violate the LPC with respect to the new spline space. If this happens, then $(E, X, \mathscr{S}_p^m)$ must violate the LPC, contrary to hypothesis.

A condition corresponding to $e_{i,0} = 1$ is called a *Lagrange condition* and we say that we are adding a Lagrange condition at t to an HBI problem $(E, X, \mathscr{S}_p^m)$ if a zero in the $j = 0$ column is changed to a one, possibly by adding a new row to E if $t \notin X$.

LEMMA 1.4. *Suppose $(E, X, \mathcal{S}_n^m)$ satisfies the LPC but is not full. Then there exists a point $t \in X \cup \{\xi_\nu\}_1^q$, where a Lagrange condition can be added to $(E, X, \mathcal{S}_n^m)$ without violating the LPC.*

Proof. If $E(0: l, s) = m - p(0: l, s)$ for some $0 < l < s \leq q - 1$ or $0 \leq l < s < q - 1$, then we can decompose according to Lemma 1.1 and consider one of the split problems which is not full. Thus without loss of generality we assume this never happens. But then *any* Lagrange condition can be added without violating the LPC.

LEMMA 1.5. *Assume that $(E, X, \mathcal{S}_n^m)$ satisfies the LPC but is not full. Without loss of generality, assume that $[\{\xi_\nu\}_1^q \cup \{b\}] \subset X$, possibly by having some rows with all zero entries in E. Then we can "fill" E in such a way that the LPC remain valid by*

(i) *changing some "boxed" zeros to minus ones,*

(ii) *changing some "boxed" ones to twos, and/or*

(iii) *changing some zeros to ones in the last row corresponding to b.*

Proof. Inductively for $l = 1, 2, \ldots, q$ we make changes of type (i) or (ii) for "boxed" entries corresponding to the interpolation point and knot ξ_l so that (1.11) will be valid for that integer after the changes are made. Further we make changes one at a time for entries with j-index as large as possible without violating the LPC. To show that this is always possible, suppose we have done this for $l = 1, 2, \ldots, l_* - 1$ (if any) and have

$$E - E(0: l, q + 1) \geq \sum_{\nu=1}^{l} R_\nu, \qquad l = 1, 2, \ldots, l_* - 1. \tag{1.19}$$

Suppose $x_{i_*} = \xi_{l_*}$. If

$$E - E(0: l_*, q + 1) \geq \sum_{\nu=1}^{l_*} R_\nu, \tag{1.20}$$

then there is no need to make any changes in the i_*th row. If $e_{i_*, j} = -1$ or 2 for all $j = m - R_{l_*}, \ldots, m - 1$, then (1.20) holds. Suppose

$$e_{i_* j_*} = 0 \text{ or } 1, \qquad \text{where} \qquad m - R_{l_*} \leq j_*, \tag{1.21}$$

$$e_{i_* j} = -1 \text{ or } 2 \qquad \text{for all} \quad j = (m - R_{l_*}), \ldots, j_* - 1 \text{ (if any)}, \tag{1.22}$$

but e_{i_*, j_*} cannot be changed to -1 or 2, respectively, without violating the LPC. By (1.10), there exist integers $\tilde{\eta}$, $\tilde{l}$, and $\tilde{s}$ with $\tilde{l} < l_* \leq \tilde{s}$, $\tilde{\eta} \leq j_*$, $R_\nu < m - \tilde{\eta}$ for all $\tilde{l} < \nu < \tilde{s}$ (if any), and

$$E(\tilde{\eta}: \tilde{l}, \tilde{s}) = m - \tilde{\eta} + p(\tilde{\eta}: \tilde{l}, \tilde{s}). \tag{1.23}$$

Then

$$
\begin{aligned}
\| E \| - \| E(0: l_*, q+1) \| & \\
\geqslant (\| E \| - \| E(0: \tilde{l}, q-1) \|) + \| E(\tilde{\eta}: \tilde{l}, \tilde{s}) \| & \\
- \left\{ \begin{array}{ll} \| E(\tilde{\eta}: l_*, \tilde{s}) \|, & \text{if } l_* < \tilde{s} \\ \max[0, m - R_{l_*} - \tilde{\eta}], & \text{if } l_* = \tilde{s} \end{array} \right\} + \max[0, \tilde{\eta} - m + R_{l_*}] & \\
\geqslant \sum_{\nu=1}^{l_*} R_\nu . & \qquad (1.24)
\end{aligned}
$$

Again (1.20) holds. Thus we can always accomplish the induction step.

If the matrix is still not full after all of these changes, then we make changes of type (iii) at entries with j-index as large as possible without violating the LPC. We argue in a similar manner that if it is not possible to change some such entry, then it is unnecessary to do so.

EXAMPLE 1.1.

$$
E = \begin{pmatrix} 0 & 0 & 0 & 0 & \boxed{0} \\ 0 & 0 & \boxed{0 \quad 2 \quad 1} \\ 0 & 0 & \boxed{0 \quad 0 \quad 0} \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

becomes via the procedure given for Lemma 1.5

$$
\hat{E} = \begin{pmatrix} 0 & 0 & 0 & 0 & \boxed{-1} \\ 0 & 0 & \boxed{-1 \quad 2 \quad 2} \\ 0 & 0 & \boxed{-1 \quad 0 \quad 0} \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}.
$$

The display E is quasipoised and $\hat{E}$ is poised by Theorem 1.1.

2. BEST L_1-APPROXIMATION BY SPLINES WITH GENERALIZED CONVEXITY CONSTRAINTS

Let integers $0 \leqslant k_0 < k_1 < \cdots < k_w \leqslant m - 1$ and $\epsilon_v = \pm 1$, $v = 0, 1, \ldots, w$ be given. Suppose $\mathscr{S}_\nu^m \subset C[a, b]$, i.e., $R_\nu < m$, $\nu = 1, \ldots, q$ and $m > 1$.

Define

$$G = \{\psi \in \mathcal{S}_p^m : \epsilon_v \psi^{(k_v)}(t) \geq 0, a \leq t \leq b, v = 0, 1, ..., w\}. \quad (2.1)$$

Recall we assume right continuity of all spline derivatives. Also for every $g \in G$ we have

$$\epsilon_v g^{(k_v)}(\xi_{v}-) \geq 0 \qquad \text{if} \quad k_v \geq m - R_v . \quad (2.2)$$

For an integrable function h, let $\|h\|_1 = \int_a^b |h(t)| \, dt$. Suppose f is in $C[a, b]$, the space of continuous functions defined on $[a, b]$, but is not in G. Then a best L_1-approximation to f from G is a spline $g_* \in G$ such that

$$\|g_* - f\|_1 = \inf_{g \in G} \|g - f\|_1 . \quad (2.3)$$

Denote by $P_G(f)$ the collection of all such best approximations. $P_G(f) \neq \emptyset$ because G is closed, convex, and finite dimensional. We have the following characterization theorem.

THEOREM 2.1. *Assume $f \in C[a, b]$ and G is defined as in (2.1). Then there exist*

(i) *functions $\varphi_1, ..., \varphi_r$, $r \geq 1$ where $|\varphi_i(t)| = 1$ for almost every $t \in [a, b]$, $i = 1, ..., r$,*

(ii) *an HBI problem $(E, X, \mathcal{S}_p^m)$, E , $\cdot r \times m - p - 1$, where $e_{ij} = 0$ only if $j = k_v$ for some v,*

(iii) *positive scalars $\lambda_1, ..., \lambda_r$, and*

(iv) *scalars $\{\mu_{ij}\}$ and $\{\mu_{ij}^-\}$ for $i = 1, ..., k$ and $j = 0, 1, ..., m - 1$, where $\mu_{ij} = 0$ if $e_{ij} = 1$ or 2 and $\mu_{ij}^- = 0$ if $e_{ij} = -1$ or 2 and where* sgn μ_{ij} = sgn μ_{ij}^- = $-$sgn ϵ_v *if $j = k_v$ with*

$$\sum_{i=1}^{r} \lambda_i \int_a^b \varphi_i(t) \, \psi(t) \, dt - \sum \mu_{ij} \psi^{(j)}(x_i)$$
$$- \sum \mu_{ij}^- \psi^{(j)}(x_{i}-) = 0, \qquad \text{for all} \quad \psi \in \mathcal{S}_p^m, \quad (2.4)$$

such that $g \in G$ is in $P_G(f)$ if and only if

$$\int_a^b \varphi_i(t)[g(t) - f(t)] \, dt = \|g - f\|_1 , \qquad i = 1, ..., r, \quad (2.5)$$

$$g^{(j)}(x_i) = 0, \qquad \text{whenever} \qquad e_{ij} = 1 \text{ or } 2, \quad (2.6)$$

and

$$g^{(j)}(x_{i}-) = 0, \qquad \text{whenever} \qquad e_{ij} = -1 \text{ or } 2. \quad (2.7)$$

Proof. This theorem is a special case of a theorem of Rozema and Smith [11, Theorem 4.1] once we note that it is easy to find a polynomial $\psi \in \mathscr{S}_p^m$ satisfying

$$\epsilon_v \psi^{(k_v)}(t) > 0; \qquad a \leqslant t \leqslant b; \quad v = 0, 1, \ldots, w. \tag{2.8}$$

LEMMA 2.1. *If $f \in C[a, b]$, g_1 and g_2 are elements of $P_G(f)$, and $g_0 = \frac{1}{2}(g_1 + g_2)$, then $g_1 - g_2$ vanishes at the zeros of $g_0 - f$.*

This well-known lemma can be found in [2] and is a special case of [11, Lemma 6.1]. In particular, Theorem 2.1 and Lemma 2.1 characterize best L_1-approximation by continuous unconstrained splines with fixed knots, i.e., when $G = \mathscr{S}_p^m$. In order to make the proof of Theorem 2.3 below more transparent, we use the same technique to prove uniqueness in the unconstrained case. This proof is quite different from those in [2, 4] where a similar result is established.

THEOREM 2.2. *For every $f \in C[a, b]$, there is a unique best L_1-approximation from $\mathscr{S}_p^m$, the linear space of polynomial splines with fixed knots ξ_ν each with multiplicities R_ν, $\nu = 1, \ldots, q$, respectively, where $\sum_{\nu=1}^q R_\nu = p$.*

Proof. Existence follows from standard arguments since $\mathscr{S}_p^m$ is closed, convex, and finite dimensional. We enumerate the steps in the proof of uniqueness for easy reference and comparison in the subsequent proof.

Step 1. Let $\varphi_1, \ldots, \varphi_r$ and $\lambda_1, \ldots, \lambda_r$ be as guaranteed by Theorem 2.1. No HBI problem and no scalars $\{\mu_{ij}\}$ and $\{\mu_{\bar{i}\bar{j}}\}$ are needed because there are no constraints.

Step 2. Suppose g_1 and g_2 are both best L_1-approximations to f from $\mathscr{S}_p^m$. Then so is $g_0 = \frac{1}{2}(g_1 + g_2)$. Considering (2.5) for g_0, we see that for almost every t with $g_0(t) \neq f(t)$ we must have

$$\varphi_i(t) = \operatorname{sgn}[g_0(t) - f(t)], \qquad i = 1, \ldots, r. \tag{2.9}$$

Let Φ be the right continuous function defined by

$$\Phi(t) = \lim_{\epsilon \searrow 0} \operatorname{sgn}\, [g_0(t + \epsilon) - f(t + \epsilon)]. \tag{2.10}$$

Define T to be the closure of the set of all points where Φ is either zero or changes sign. Thus $g_0(t) = f(t)$ for every $t \in T$ (and possibly at other isolated points where $g_0 - f$ does not change sign).

Step 3. Do nothing here since there are no constraints.

Step 4. Choose a *maximal* subset $\{t_1 < t_2 < \cdots < t_\alpha\}$ from T with the property that Lagrange interpolation at these points is quasi-poised. This property is equivalent to asking that point evaluations at these points be linearly independent functionals in the dual of $\mathcal{S}_p^m$ or that some extension of this set satisfies the Schoenberg–Whitney interlacing condition. Let $(E_1, X_1, \mathcal{S}_p^m)$ denote this Lagrange interpolation problem.

Step 5. We claim that $\alpha = |E_1| = m - p$. If not, then there exists a point $t_* \in [a, b]$ which is not one of the points $t_1, \ldots, t_\alpha$ nor one of the knots $\xi_1, \ldots, \xi_q$ with the property that a Lagrange condition at t_* may be added to $(E_1, X_1, \mathcal{S}_p^m)$ and the resulting interpolation problem is still quasi-poised. This is a simple application of Lemma 1.4. This point t_* cannot be from T because $t_1, \ldots, t_\alpha$ was a maximal subset of T with this property. Let $(E_2, X_2, \mathcal{S}_p^m)$ denote $(E_1, X_1, \mathcal{S}_p^m)$ with this Lagrange condition at t_* added.

Now we use $(E_2, X_2, \mathcal{S}_p^m)$ to construct a spline $\psi_* \in \mathcal{S}_p^m$ having sign structure similar to Φ and contradicting (2.4). If $\alpha + 1 = m - p$, this can be done immediately. When $\alpha + 1 < m - p$, the typical technique in spline proofs of this type would be to construct ψ_* with lower knot multiplicities or fewer knots (possibly even no knots at all if $\alpha + 1 \leq m$). Since this technique does not generalize easily to the constrained case, we use the following alternative.

Without loss of generality, assume that the points $\xi_1, \ldots, \xi_q$, and b are included in the set of nodes X_2, possibly by introducing zero rows in the incidence matrix E_2. Inductively for $l = 1, 2, \ldots, q$, make the following changes in the incidence matrix. If $x_{i_*} = \xi_l$ and

$$\sum_{i=1}^{i_*} \sum_{j=0}^{m-1} e_{ij} = \sum_{\nu=1}^{l} R_\nu - \eta, \tag{2.11}$$

where $\eta > 0$, change $e_{i_*, m-\eta}, e_{i_*, m-\eta+1}, \ldots$, and $e_{i_*, m-1}$ from zero to minus one. If $\eta \leq 0$, make no changes and continue to the next integer l. After these inductive changes have been made, if

$$\sum_{i=1}^{k} \sum_{j=0}^{m-1} e_{ij} = m - p - \eta, \tag{2.12}$$

with $\eta > 0$, where $x_k = b$, change $e_{k, m-\eta}, e_{k, m-\eta+1}, \ldots$, and $e_{k, m-1}$ from zero to one. This procedure is exactly the one described in Lemma 1.5 where the proof is given that it is always possible to do so. Denote the resulting full HBI problem thus defined by $(\bar{E}, \bar{X}, \mathcal{S}_p^m)$. This problem is poised by Theorem 1.1.

There exists a unique $\psi_* \in \mathscr{S}_p^m$ with

$$\begin{aligned} &\psi_*(t_*) = \Phi(t_*) \neq 0, \\ \psi_*^{(j)}(\bar{x}_i) = 0, \quad &\text{whenever} \quad \bar{e}_{ij} = 1 \text{ or } 2 \text{ and } \bar{x}_i \neq t_*, \\ \psi_*^{(j)}(\bar{x}_{i-}) = 0, \quad &\text{whenever} \quad \bar{e}_{ij} = -1 \text{ or } 2. \end{aligned} \tag{2.13}$$

Note that the changes made by using Lemma 1.5 above effectively lowered the degree of ψ_* on some knot intervals. (2.13) explicitly requires that $\psi_*(t_i) = 0$ for $i = 1,..., \alpha$. If $t \in T\setminus\{t_1 ,..., t_\alpha\}$, then the reason t could not be added originally to $\{t_1 ,..., t_\alpha\}$ must be that t is in some $[\xi_l , \xi_s]$ where equality occurred in the local Polya condition indexed by $(0{:}\ l, s)$ for $(E_1 , X_1 , \mathscr{S}_p^m)$, hence for $(\bar{E}, \bar{X}, \mathscr{S}_p^m)$. Further since E_1 is already "full" on $[\xi_l , \xi_s]$ then $t_* \notin [\xi_l , \xi_s]$. Decomposing $(\bar{E}, \bar{X}, \mathscr{S}_p^m)$ using Lemma 1.1 yields a split poised problem on $[\xi_l , \xi_s]$ with only zero data values from (2.13). Thus ψ_* is identically zero on $[\xi_l , \xi_s]$ and $\psi_*(t) = 0$ for all $t \in T$.

Similarly, if $|\bar{E}(0{:}\ l, s)| = m + p(0{:}\ l, s)$ for some $0 \leqslant l < s \leqslant q - 1$ where $t_* \notin [\xi_l , \xi_s]$, then the problem decomposes according to Lemma 1.1. Examining the part of (2.13) which each split problem must satisfy, we conclude that $\psi_*(t) = 0$ for all $t \in [a, \xi_s]$ if $\xi_s < t_*$ or $\psi_*(t) = 0$ for all $t \in [\xi_l , b]$ if $t_* < \xi_l$. If $\bar{x}_i = \xi_\nu = t_*$ and $\bar{e}_{ij} = 1$ for all $j = 0, 1,..., m - R_\nu - 1$, then $(\bar{E}, \bar{X}, \mathscr{S}_p^m)$ can be decomposed according to Lemma 1.2. As above, the split problem not involving t_* will be homogeneous so that ψ_* will be identically zero either for all $t < \bar{x}_i$ or for all $t > \bar{x}_i$, but not both.

Suppose $\psi_*(t) = 0$ for all $t \in [\xi_{l_0} , \xi_{s_0}]$. For some $0 \leqslant l_0 < s_0 \leqslant q + 1$, $t_* \notin [\xi_{l_0} , \xi_{s_0}]$. Let $(E^*, \bar{X}, \mathscr{S}_p^m)$ denote $(\bar{E}, \bar{X}, \mathscr{S}_p^m)$ with the Lagrange condition at t_* deleted. Let t_0 be any point from (ξ_{l_0} , ξ_{s_0}) but not in $\bar{X}$. A Lagrange condition at t_0 cannot be added to $(E^*, \bar{X}, \mathscr{S}_p^m)$ without violating the LPC. If it could be, it would give a full poised HBI problem for which the nontrivial ψ_* satisfies all zero data values, which is impossible. Thus there exist integers $0 \leqslant l < s \leqslant q + 1$ with $\xi_l \leqslant t_0 \leqslant \xi_s$, $t_* \notin [\xi_l , \xi_s]$ and $|E^*(0{:}\ l, s)| = m + p(0{:}\ l, s)$, hence $|E(0{:}\ l, s)| = m + p(0{:}\ l, s)$. As before, ψ_* is identically zero either on $[a, \xi_{s_0}]$ if $\xi_{s_0} < t_*$ or on $[\xi_{l_0} , b]$ if $t_* < \xi_{l_0}$.

We conclude that ψ_* must be identically zero except on some knot interval $[\xi_{l_*} , \xi_{s_*}]$ containing t_* (possibly $[a, b]$). On this interval there are only a finite number of points from T (all of which are included in $\bar{X}$) and only a finite number of points where ψ_* is zero. Any sequence of $\bar{E}$ beginning with $\bar{e}_{i,0}$ for which $\xi_{l_*} < \bar{x}_i < \xi_{s_*}$ and $\bar{x}_i \neq t_*$ is strongly regular. Further, if $0 \leqslant l < s \leqslant q - 1$ and $(\xi_l , \xi_s) \cap (\xi_{l_*} , \xi_{s_*}) = \varnothing$,

$$|E^*(0{:}\ l, s)| < m + p(0{:}\ l, s). \tag{2.14}$$

Suppose ψ_* changes sign at some $t_0 \in (\xi_{l_*} , \xi_{s_*})$, where Φ does not change sign. First $t_0 \neq t_*$ because ψ_* is continuous and $\psi_*(t_*) \neq 0$. Further $t_0 \notin T$

because, at the isolated points from T in (ξ_{l_*}, ξ_{s_*}), the right continuous function Φ must change sign. By construction, if $\bar{e}_{i,0} = 1$, then either $\bar{x}_i \in T$ or $\bar{x}_i = b$. Thus a Lagrange condition at t_0 can be added to $(E^*, \bar{X}, \mathscr{S}_p^m)$, giving a HBI problem which will be full and poised by Theorem 1.1 and (2.14) together with the properties of $(E^*, \bar{X}, \mathscr{S}_p^m)$. But the nontrivial ψ_* satisfies this problem with all zero data values, which is impossible. Thus ψ_* cannot change sign when Φ does not.

The right continuous function Φ changes sign exactly at the isolated points from T which lie in (ξ_{l_*}, ξ_{s_*}). Any such point $x_i \in T \cap (\xi_{l_*}, \xi_{s_*})$ belongs to $\bar{X}$, and the corresponding entry $e_{i,0} = 1$ so that $\psi_*(x_i) = 0$. Suppose that ψ_* does not change sign at this x_i. By construction, $e_{i,0}$ begins an odd sequence $1 = e_{i,0} = e_{i,1} = \cdots = e_{i,n}$, where $e_{i,n-1} = 0$ and is not "boxed." ψ_* must have an even zero at x_i so as to not change sign so that $\psi^{(n-1)}(x_i) = 0$. Changing $e_{i,n-1}$ from zero to one in $(E^*, \bar{X}, \mathscr{S}_p^m)$ gives a HBI problem which will be full and poised by Theorem 1.1 and (2.14). Again the nontrivial ψ_* satisfies this homogeneous problem giving a contradiction.

We conclude that $\operatorname{sgn} \psi_*(t) = \Phi(t)$ at all points t where $\psi_*(t) = 0$. Therefore

$$\int_a^b \varphi_i(t)\, \psi_*(t)\, dt = \int_a^b |\psi_*(t)|\, dt > 0, \qquad i = 1,\ldots, r. \tag{2.15}$$

This with (2.13) shows that ψ_* is a spline from $\mathscr{S}_p^m$ which contradicts (2.4). Thus our claim at the beginning of Step 5 must be true.

Step 6. Then $(E_1, X_1, \mathscr{S}_p^m)$ is a poised HBI problem. By the construction of T in Step 2 and Lemma 2.1, $g_1 - g_2$ vanishes at every $t \in T$, hence $g_1 - g_2$ satisfies all zero data for $(E_1, X_1, \mathscr{S}_p^m)$. Thus by the uniqueness of poised HBI, $g_1 = g_2$. Since g_1 and g_2 were two arbitrary best approximations, the proof of uniqueness is now complete.

We now turn to best L_1-approximation from G, the set of splines satisfying certain generalized convexity constraints. Only significant differences between the proof that follows and the previous unconstrained proof will be explained in great detail.

THEOREM 2.3. *For every $f \in C[a, b]$, there is a unique best L_1-approximation from G defined as in* (2.1).

Proof. Existence again follows from standard arguments.

Step 1. Let $\varphi_1, \ldots, \varphi_r$, $\lambda_1, \ldots, \lambda_r$, $(E, X, \mathscr{S}_p^m)$, $\{\mu_{ij}\}$, and $\{\mu_{\bar{i}\bar{j}}\}$ be as guaranteed by Theorem 2.1. Without loss of generality we may assume that $(E, X, \mathscr{S}_p^m)$ is quasi-poised because any dependency in these conditions could be used to accomplish (2.4), (2.6), and (2.7) with a smaller HBI problem made up of independent conditions.

Step 2. This step is exactly the same as in the proof of the previous Theorem. Namely, if g_1 and g_2 are both in $P_G(f)$, then so is $g_0 = \frac{1}{2}(g_1 + g_2)$. Φ and T are defined as before.

Step 3. Suppose $g \in P_G(f)$. If $e_{ij} = 1$, then $j = k_v$ for some v and $g^{(j)}(x_i) = 0$ by (2.6). By the definition of G in (2.1), $g^{(j)}$ does not change sign. Using the zero counting procedure for splines devised by Schumaker [12] (see also [7, 8]), this must mean that x_i is in some interval (possibly the point alone) where $g^{(j)}$ is identically zero and that this interval is either an even zero for $g^{(j)}$ or it contains one of the endpoints a or b. Thus if $a < x_i < b$, $j < m - 1$, $e_{ij} = 1$, and $e_{i,j+1}$ is not "boxed," i.e., no spline in $\mathscr{S}_p^m$ may have a discontinuity in its $(j+1)$-derivative at x_i, then

$$g^{(j-1)}(x_i) = 0. \tag{2.16}$$

One by one, change "unboxed" zeros to ones in E to assure that there are no odd strongly regular sequences in rows for which $a < x_i < b$ and, if the sequence begins with $e_{i,0} = 1$, for which $x_i \notin T$. If $x_i \in T$ and $e_{i,0} = 1$, we leave the order one sequence odd. By (2.16), any $g \in P_G(f)$ will be zero for any such added condition. If it is necessary to preserve the LPC, a simple knot, not already an interpolation point in X, is added to the spline space as described in Lemma 1.3 when a change is made in E.

Let $(\tilde{E}, X, \mathscr{S}_{\tilde{p}}^m)$ denote the resulting HBI problem, which will be quasi-poised by Theorem 1.1, where $\mathscr{S}_p^m \subseteq \mathscr{S}_{\tilde{p}}^m$ and $p \leqslant \tilde{p}$ because of the possible addition of simple knots. Now (2.4) may not hold for the space $\mathscr{S}_{\tilde{p}}^m$. However, it can be shown using elementary linear algebra that the linear dependencies of conditions requiring the addition of simple knots (i.e., the violation of the LPC) together with (2.4) imply that there exist scalars $\{\tilde{\mu}_{ij}\}$ and $\{\tilde{\mu}_{\bar{ij}}\}$ having no particular sign convention, with $\tilde{\mu}_{ij} = 0$ if $\tilde{e}_{ij} \neq 1$ or 2 and $\tilde{\mu}_{\bar{ij}} = 0$ if $\tilde{e}_{ij} \neq -1$ or 2, such that

$$\sum_{i=1}^{r} \lambda_i \int_a^b \varphi_i(t)\, \psi(t)\, dt + \sum_{i,j} \tilde{\mu}_{ij} \psi^{(j)}(x_i) + \sum_{i,j} \tilde{\mu}_{\bar{ij}} \psi^{(j)}(x_{i^-}) = 0 \qquad \text{for all} \quad \psi \in \mathscr{S}_{\tilde{p}}^m. \tag{2.17}$$

For any $g \in P_G(f) \subset \mathscr{S}_p^m \subset \mathscr{S}_{\tilde{p}}^m$ (in particular for g_1 and g_2) and $j > 0$,

$$g^{(j)}(x_i) = 0 \qquad \text{whenever} \qquad \tilde{e}_{ij} = 1 \text{ or } 2, \tag{2.18}$$

and

$$g^{(j)}(x_{i^-}) = 0 \qquad \text{whenever} \qquad \tilde{e}_{\bar{ij}} = -1 \text{ or } 2. \tag{2.19}$$

Step 4. Choose a *maximal* subset $\{t_1 < t_2 < \cdots < t_\alpha\}$ from $T \backslash \{x_i : \tilde{e}_{i,0} = 1\}$ with the property that Lagrange interpolation conditions at

$t_1, \ldots, t_s$ can be added to $(\tilde{E}, X, \mathcal{S}_{\tilde{p}}^{m})$ giving $(E_1, X_1, \mathcal{S}_{\tilde{p}}^{m})$ without violating the LPC. This new $(E_1, X_1, \mathcal{S}_{\tilde{p}}^{m})$ still satisfies C because $(\tilde{E}, X, \mathcal{S}_{\tilde{p}}^{m})$ did by construction in Step 3. so by Theorem 1.1 both will be quasi-poised.

Step 5. We claim that $|E_1| = m + \tilde{p}$. If not, then we arrive at a contradiction by constructing $\psi_* \in \mathcal{S}_{\tilde{p}}^{m}$ which violates (2.17) in a manner almost identical to the way an element of $\mathcal{S}_{p}^{m}$ was constructed in the corresponding step of the previous proof which violated (2.4). One slight difference is that the application of Lemma 1.5 will be somewhat more complicated than (2.11) and (2.12) because E_1 may now have nonzero entries in the $j = 0$ columns. The conclusion is the same after the application, however.

Also the examination of the sign-changing properties may be somewhat more complicated but the conclusion will remain valid because of the construction of $\tilde{E}$ in Step 3.

Step 6. Then $(E_1, X_1, \mathcal{S}_{\tilde{p}}^{m})$ is a poised HBI problem for which $g_1 - g_2$ satisfies zero data by (2.18), (2.19), and Lemma 2.1 together with the fact that for every $t \in T$, $g_0(t) = f(t)$. Thus $g_1 = g_2$ and the proof of the theorem is complete.

3. Monotonicity and Convexity

Definition (2.1) for G in the previous section is a natural generalization to splines of the notion of monotone polynomials introduced by Shisha [13] (see also [5]). Included is the possibility for requiring nonnegativity or nonpositivity by choosing $k_0 = 0$. Since we made the assumption that all of the elements of our spline space $\mathcal{S}_{p}^{m}$ were continuous, choosing some $k_r = 1$ requires the usual monotonicities, either nondecreasing or nonincreasing. If $R_\nu < m - 1$, $\nu = 1, \ldots, q$, then some $k_r = 2$ implies that all of the elements of G are either convex or concave. However it is reasonable to ask for convexity or concavity even if some of the knots have multiplicity $m - 1$. Convexity is well defined (although not in terms of the second derivative) for linear splines, i.e., continuous piecewise linear functions, for example. Similarly monotonicity is well defined for discontinuous splines.

We briefly indicate how the preceding section would need to be modified to include the requirement of convexity when $R_\nu = m - 1$ for some of $\nu = 1, \ldots, q$. We ask that

$$\psi^{(2)}(t) \geq 0, \qquad a \leq t \leq b, \tag{3.1}$$

and

$$\psi^{(1)}(\xi_{\nu+}) - \psi^{(1)}(\xi_{\nu-}) \geq 0 \qquad \text{whenever} \qquad R_\nu = m - 1. \tag{3.2}$$

The conditions in (3.2) are also linear constraints on $\mathcal{S}_p^m$. It is not difficult to show that there exists a spline in $\mathcal{S}_p^m$ which satisfies all of the constraints including these strictly so that the theorem of Rozema and Smith [11, Theorem 4.1] applies.

If none of the constraints of type (3.2) are chosen by the theorem of Rozema and Smith, then we proceed exactly as in the previous section. On the other hand if one of these "jump" constraints is active and is chosen, that implies that for $g \in P_G(f)$,

$$g^{(1)}(\xi_{\nu}-) = g^{(1)}(\xi_{\nu}-), \tag{3.3}$$

i.e., the knot ξ_ν is really only of multiplicity $m - 2$ for all splines in $P_G(f)$. It is easy to show that $P_G(f) \subset P_{G \cap \mathcal{S}_{p'}^m}(f)$, where $\mathcal{S}_{p'}^m$ is $\mathcal{S}_p^m$ with the knots chosen in (3.3) having multiplicity only $m - 2$ so that $p' < p$. In fact the above inclusion is an equality and $P_{G \cap \mathcal{S}_{p'}^m}(f)$ can be characterized using the arguments of the previous section. In particular we can still conclude that uniqueness holds.

4. Further Extensions

With only minor modifications the work of this paper can be extended to the problem of finding a best global L_1-approximation to a compact (in $L_1[a, b]$) set of continuous functions F from G as defined in (2.1). Such best global approximations are also called restricted Chebyshev centers for F with respect to G or best approximations to the elements of F simultaneously. The methods of the paper by Rozema and Smith [11] apply in a straightforward manner.

The techniques we have used can also be applied to the more general problem of best L_1-approximation by splines with restricted ranges of their derivatives. In the uniform norm, the corresponding polynomial problem was introduced by Roulier and Taylor [10] and the spline problem was studied in [7]. We wish to point out the significant differences between uniform and L_1-approximation by these restricted splines.

Examining the proofs in [10, 7], only the functions which bound the ranges of the derivatives (other than the zero derivative) need to be assumed to be differentiable in order to guarantee uniqueness in the uniform norm. In L_1, the functions which bound the range need to be differentiable as well in order to carry out the part of Step 5 where it is shown that the constructed ψ_* does change sign at all isolated $t \in T$ but does not change sign at x_i where $x_i \notin T$ and $e_{i,0} = 1$. Thus where we made sure in Step 3 that when $e_{i,0} = 1$ and $x_i \notin T$ we had an even sequence, we were using the fact that the bounding function zero on the range was differentiable. The problem of one-sided

L_1-approximation of a differentiable function by splines which was studied by Pinkus [9] is a special case where the given function is also the range bound.

If there are bounds on the range, then for uniqueness in the uniform norm to be assured, the assumption is needed that the given function satisfies these range bounds at least within some $\epsilon > 0$, where ϵ is strictly less than the distance from the given function to the set of restricted splines. If this is not the case, then a single linear functional (a point evaluation) in $C^*[a, b]$ may be a positive error-extremal and a negative constraint-extremal or vice in the terminology of [7]. No such assumption is needed ni L_1 although the assumption that the given function is continuous is needed.

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